Motivation:

- inner product $\langle , \rangle \in V^* \otimes V^*$ $\Longrightarrow$ For subspaces $K, L \leq V$, $K \perp_{\langle,\rangle} L$ (i.e. $\langle v, w \rangle = 0$ $\forall\, v \in K$ and $w \in L$) implies $K \cap L = \{0\}$.

  ↑
  non-deg, <u>symmetric</u>, non-negative

- "skew" product $\omega \in V^* \otimes V^*$ $\Longrightarrow$ $\exists$ non-trivial $L \leq V$ s.t. $L \perp_\omega L$ (i.e. $\omega(v, w) = 0$ $\forall\, v, w \in L$)

  ↑
  non-deg, <u>anti-symmetric</u> (valued in $\mathbb{R}$)

e.g. $V = \mathbb{R}^2 = \operatorname{span}\{e_1, e_2\}$ $L = \operatorname{span}\{e_1\}$, ω any "skew" product.

Then $\omega(\underbrace{\lambda e_1}_{v}, \underbrace{\eta e_1}_{w}) = \lambda \eta\, \omega(e_1, e_1) = 0$

e.g. $\omega(v_1 e_1 + v_2 e_2, w_1 e_1 + w_2 e_2) := \det \begin{pmatrix} v_1 & w_1 \\ 2 & w_2 \end{pmatrix}$

So, there are essential differences between symmetric and anti-symmetric 2-tensors.

Denote by S_k the k-th symmetric group, associate

$$\sigma \in S_k \rightsquigarrow \mathrm{sgn}(\sigma) = \begin{cases} -1 & \text{if } \sigma \text{ is an odd permutation} \\ 1 & \text{if } \sigma \text{ is an even permutation} \end{cases}$$

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} \in S_5$ $\quad \mathrm{sgn}(\sigma) = -1$ b/c $\sigma = (12)(13)(45)$

$$\begin{tikzcd}
 & \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \ (= V^{*,\otimes k}) \arrow[dl, "\text{Sym (对称化)}"'] \arrow[dr, "\text{Alt (反对称化)}"] & \\
\Sigma^k V^* := \{ T \in V^{*,\otimes k} \mid \sigma \cdot T = T \} & & \Lambda^k V^* := \{ T \in V^{*,\otimes k} \mid \sigma \cdot T = \mathrm{sgn}(\sigma) \cdot T \}
\end{tikzcd}$$

$$(\sigma \cdot T)(v_1, \cdots, v_k) := T(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

$$\mathrm{Sym}(T) := \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \cdot T \qquad \mathrm{Alt}(T) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) (\sigma \cdot T)$$

e.g. $V^{*,\otimes 2}$ $\quad v\otimes w \longrightarrow \mathrm{Sym}(v\otimes w) = \frac{1}{2}(v\otimes w + w\otimes v)$

$v\otimes w \longrightarrow \mathrm{Alt}(v\otimes w) = \frac{1}{2}(v\otimes w - w\otimes v)$

Note that $v\otimes w = \mathrm{Sym}(v\otimes w) + \mathrm{Alt}(v\otimes w)$. $(*)$

<u>Rmk.</u> The relation $(*)$ is misleading - most cases this fails.

e.g. $V^* = (\underset{\substack{\uparrow \\ \mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\}}}{\mathbb{R}^3})^*$ and consider dual basis $\{e^1, e^2, e^3\}$ of $(\mathbb{R}^3)^*$

and $e^1\otimes e^2\otimes e^3 \in V^{*,\otimes 3}$. Suppose $e^1\otimes e^2\otimes e^3 = \underset{\Sigma^3 V^*}{\overset{\cap}{a}} + \underset{\Lambda^3 V^*}{\overset{\cap}{b}}$

Then

$$(e^1\otimes e^2\otimes e^3)(e_1,e_2,e_3) = 1 = a(e_1,e_2,e_3) + b(e_1,e_2,e_3)$$

$$(e^1\otimes e^2\otimes e^3)(e_2,e_3,e_1) = 0$$

$\sigma = (1,2)(1,3)$

$$\overset{\shortparallel}{a(e_2, e_3, e_1) + b(e_2,e_3,e_1)} = (\sigma\cdot a)(e_1,e_2,e_3) + (\sigma\cdot b)(e_1,e_2,e_3)$$

$$= a(e_1,e_2,e_3) + (+1)\, b(e_1,e_2,e_3) \ (=1) \ \rightarrow\leftarrow.$$

Both $\Sigma^k V^*$ and $\Lambda^k V^*$ are interesting, but let us focus on $\Lambda^k V^*$.

elements inside are called <u>alternating tensors</u>

- $\bigoplus_k \Lambda^k V^*$ is an associative super-commutatitive algebra ← <u>exterior algebra</u>

- $$\underset{\Lambda^k V^*}{\alpha}, \underset{\Lambda^\ell V^*}{\beta} \xmapsto{\wedge} \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\alpha \otimes \beta) \in \Lambda^{k+\ell} V^*$$

wedge product

$$\left(= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma) \left(\sigma \cdot \alpha \otimes \beta\right) \right)$$

- $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

- $\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$ ($\Rightarrow$ if one of α, β lies in $\Lambda^{\text{even}} V^*$, then $\alpha \wedge \beta = \beta \wedge \alpha$)

<u>Rmk</u> $\bigoplus_{\text{even}} \Lambda^{\text{even}} V^*$ is an associative and commutatitive (sub) algebra

e.g. $\alpha, \beta \in \Lambda^1 V^*$, then

$$(\alpha \wedge \beta)(v, w) = (\alpha \otimes \beta)(v, w) - (\beta \otimes \alpha)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v).$$

e.g. $\alpha \in \Lambda^1 V^*$ and $\beta \in \Lambda^2 V^*$, then

$$2(\alpha\wedge\beta)(u,v,w) = \underbrace{(\alpha\otimes\beta)(u,v,w)}_{\sigma=(1)} - \underbrace{(\alpha\otimes\beta)(v,u,w)}_{\sigma=(1,2)} - \underbrace{(\alpha\otimes\beta)(w,v,u)}_{\sigma=(1,3)}$$

$$- \underbrace{(\alpha\otimes\beta)(u,w,v)}_{\sigma=(2,3)} + \underbrace{(\alpha\otimes\beta)(v,w,u)}_{\sigma=(1,2)(1,3)} + \underbrace{(\alpha\otimes\beta)(w,u,v)}_{\sigma=(1,3)(2,3)}$$

<u>Exe</u> Suppose $\{e^1, \cdots, e^n\}$ is a basis of V^*, then

$$\{ e^{i_1}\wedge\cdots\wedge e^{i_k} \mid 1 \le i_1 < \cdots < i_k \le n \}$$

form a basis of $V^{*,\otimes k}$. Therefore $\dim V^{*,\otimes k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. //

$\Rightarrow$ ① $\underbrace{V^*\wedge\cdots\wedge V^*}_{>n} = 0$

$\Rightarrow$ ② $\underbrace{V^*\wedge\cdots\wedge V^*}_{=n}$ has $\dim = 1$ $(= \operatorname{span}\{e^1\wedge\cdots\wedge e^n\})$

where $(e^1\wedge\cdots\wedge e^n)(v_1, \cdots, v_n) = \det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

vectors